17. Triple Integrals Part 2, Change of Variables

In this lecture, we will discuss

- Triple Integrals in Cylindrical and Spherical Coordinates, Part 1
- Change of Variables in Multiple Integrals
    - Change of Variables in Double Integrals
        - Jacobian of the transformation
        - Application: Revisit the Change to Polar Coordinates in a Double Integral
    - Change of Variables in Triple Integrals
        - Jacobian of the transformation
        - Application: Revisit the Change to Spherical Coordinates in a Triple Integral

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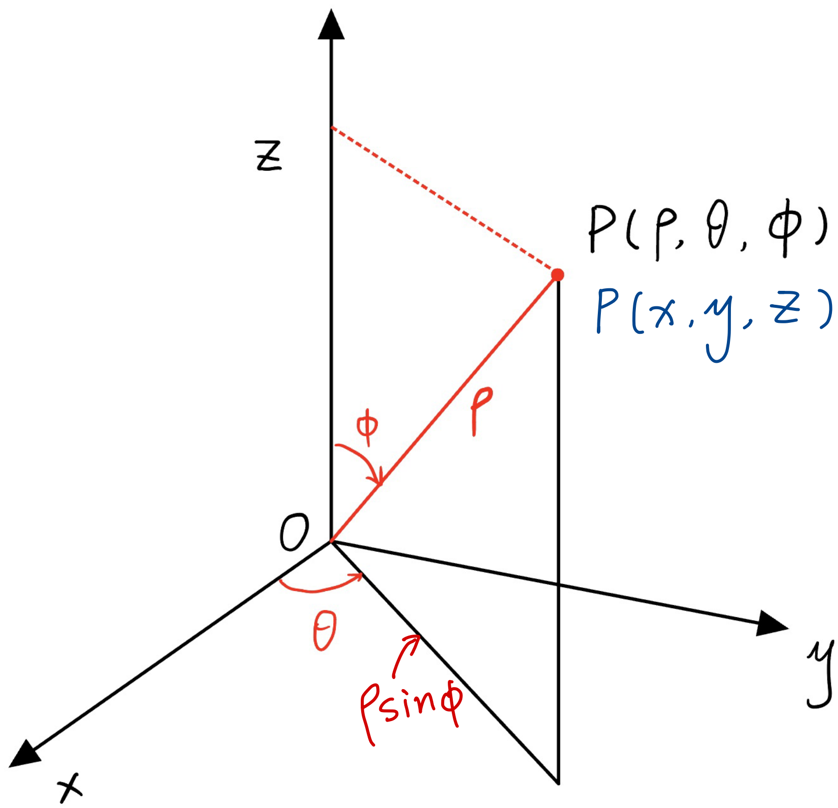
Spherical Coordinates

Recall from Lecture 7, we define the spherical coordinates (ρ, θ, ϕ) of a point (see the figure below). The relationships between the sperical coordinates and the rectangular coordinates are:

$$x = \rho \sin\phi \cos\theta \quad y = \rho \sin\phi \sin\theta \quad z = \rho \cos\phi$$



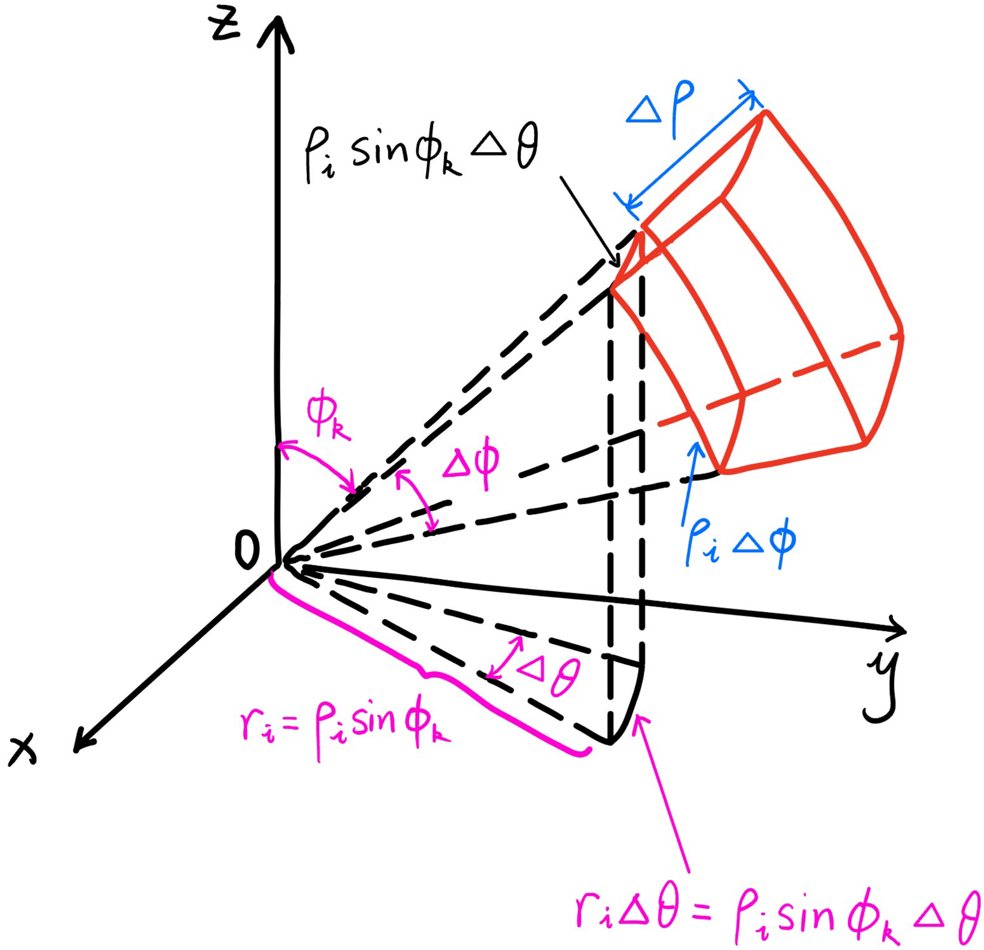
The spherical coordinates of a point

- In the spherical coordinate system the counterpart of a rectangular box is a spherical wedge

$$E = \{(\rho, \theta, \phi) \mid a \leqslant \rho \leqslant b, \alpha \leqslant \theta \leqslant \beta, c \leqslant \phi \leqslant d\}$$

  where $a \geqslant 0$ and $\beta - \alpha \leqslant 2\pi$.

- It can be shown that dividing a solid into small spherical wedges always gives the same result as dividing it into small boxes.
- So we divide E into smaller spherical wedges E_{ijk} by means of equally spaced spheres $\rho = \rho_i$, half-planes $\theta = \theta_j$, and half-cones $\phi = \phi_k$ as the figure below.



- The above figure shows that E_{ijk} is approximately a rectangular box with dimensions $\Delta\rho$, $\rho_i\Delta\phi$ (arc of a circle with radius ρ_i, angle $\Delta\phi$), and $\rho_i \sin\phi_k \Delta\theta$ (arc of a circle with radius $\rho_i \sin\phi_k$, angle $\Delta\theta$).

- So an approximation to the volume of E_{ijk} is given by

$$(\Delta\rho) \times (\rho_i \Delta\phi) \times (\rho_i \sin\phi_k \Delta\theta) = \rho_i^2 \sin\phi_k \Delta\rho\Delta\theta\Delta\phi$$

- Thus, an approximation to a typical triple Riemann sum is

$$\sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f\left(\rho_i \sin\phi_k \cos\theta_j, \rho_i \sin\phi_k \sin\theta_j, \rho_i \cos\phi_k\right) \rho_i^2 \sin\phi_k \Delta\rho\Delta\theta\Delta\phi$$

- Note this sum is a Riemann sum for the function

$$F(\rho, \theta, \phi) = f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi) \cdot \rho^2 \sin\phi$$

- Thus we have the following formula

Theorem. Formula for Triple Integration in Spherical Coordinates

$$\begin{aligned}&\iiint_E f(x,y,z)dV\\&=\int_c^d\int_\alpha^\beta\int_a^b f(\rho\sin\phi\cos\theta,\rho\sin\phi\sin\theta,\rho\cos\phi)\rho^2\sin\phi d\rho d\theta d\phi\end{aligned}\tag{1}$$

where E is a spherical wedge given by

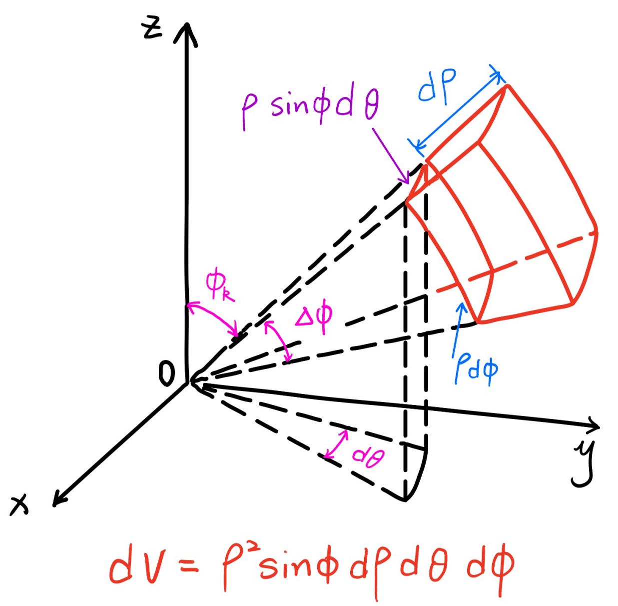
$$E=\{(\rho,\theta,\phi)\mid a\leqslant\rho\leqslant b,\alpha\leqslant\theta\leqslant\beta,c\leqslant\phi\leqslant d\}$$

Remark.

- Equation (1) states that we can convert a triple integral from rectangular coordinates to spherical coordinates by writing

$$x=\rho\sin\phi\cos\theta\quad y=\rho\sin\phi\sin\theta\quad z=\rho\cos\phi$$

  using the appropriate limits of integration, and replacing dV by $\rho^2\sin\phi d\rho d\theta d\phi$. This is explained by the figure below.



Volume Element in Spherical Coordinates

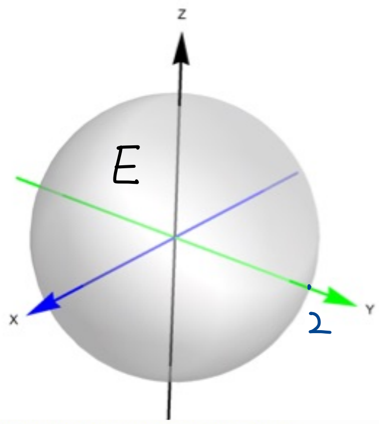
- More genearally, we can have the sperical regions like

$$E=\{(\rho,\theta,\phi)\mid\alpha\leqslant\theta\leqslant\beta,c\leqslant\phi\leqslant d,g_1(\theta,\phi)\leqslant\rho\leqslant g_2(\theta,\phi)\}$$

  In this case the formula is the same as in Equation (1) except that the limits of integration for ρ are $g_1(\theta,\phi)$ and $g_2(\theta,\phi)$.

- Usually, spherical coordinates are used in triple integrals when surfaces such as cones and spheres form the boundary of the region of integration.

Example 1. Use spherical coordinates to evaluate the triple integral $\iiint_{\mathbf{E}} x^2+y^2+z^2 dV$, where $\mathbf{E}$ is the ball: $x^2+y^2+z^2 \le 4$.



Theorem. Formula for Triple Integration in Spherical Coordinates

$$\iiint_E f(x,y,z)dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2 \sin\phi \, d\rho \, d\theta \, d\phi \quad (1)$$

where E is a spherical wedge given by

$$E = \{(\rho,\theta,\phi) \mid a \leqslant \rho \leqslant b, \alpha \leqslant \theta \leqslant \beta, c \leqslant \phi \leqslant d\}$$

ANS: The solid E can be described by spherical coordinates

$$E = \{(\rho,\theta,\phi) \mid 0 \le \rho \le 2,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi\}$$

Thus by Eq(1), we have.

$$\iiint_E x^2+y^2+z^2 \, dV$$

$$= \int_0^\pi \int_0^{2\pi} \int_0^2 \left((\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2 + (\rho\cos\phi)^2 \right) \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$= \int_0^\pi \int_0^{2\pi} \int_0^2 \rho^2 \cdot \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$= \int_0^\pi \int_0^{2\pi} \int_0^2 \rho^4 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$= \int_0^\pi \int_0^{2\pi} \frac{1}{5}\rho^5 \sin\phi \Big|_{\rho=0}^{\rho=2} d\theta \, d\phi$$

$$= \frac{2^5}{5} \int_0^\pi \int_0^{2\pi} \sin\phi \, d\theta \, d\phi$$

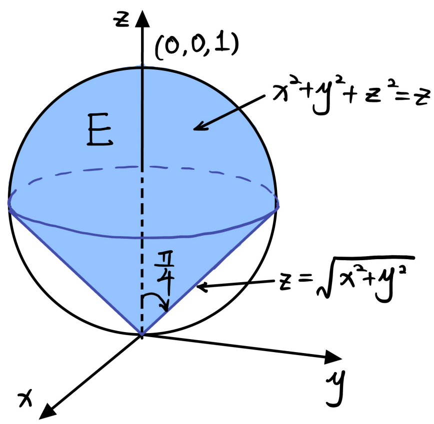
$$= \frac{2^5}{5} \int_0^{2\pi} 1 \, d\theta \int_0^\pi \sin\phi \, d\phi$$

$$= \frac{2^5}{5} \cdot \theta \Big|_0^{2\pi} \cdot [-\cos\phi] \Big|_0^\pi$$

$$= \frac{2^5}{5} \cdot 2\pi \cdot 2$$

$$= \frac{2^7}{5} \pi$$

Example 2. Use spherical coordinates to find the volume of the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$.



ANS: Denote the solid by E

$$x^2 + y^2 + z^2 = z$$

$$\Rightarrow x^2 + y^2 + z^2 - z + \frac{1}{4} = \frac{1}{4}$$

$$\Rightarrow x^2 + y^2 + \left(z - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2$$

Thus $x^2+y^2+z^2 = z$ is a sphere centered at $(0, 0, \frac{1}{2})$ with radius $\frac{1}{2}$.

In spherical coordinates, the sphere can be written as

$$x^2+y^2+z^2 = \rho^2 = z = \rho\cos\phi \Rightarrow \rho = \cos\phi$$

Also, the equation of the cone can be written as

$$z = \sqrt{x^2+y^2}$$

$$\Rightarrow \rho\cos\phi = \sqrt{(\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2} = \sqrt{\rho^2\sin^2\phi}$$

$$= \rho\sin\phi$$

$$\Rightarrow \rho\cos\phi = \rho\sin\phi$$

This implies $\cos\phi = \sin\phi \Rightarrow \phi = \frac{\pi}{4}$

Thus the solid E can be described by the spherical coordinates as

$$0 \le \theta \le 2\pi$$
$$0 \le \phi \le \frac{\pi}{4}$$
$$0 \le \rho \le \cos\phi$$

Thus

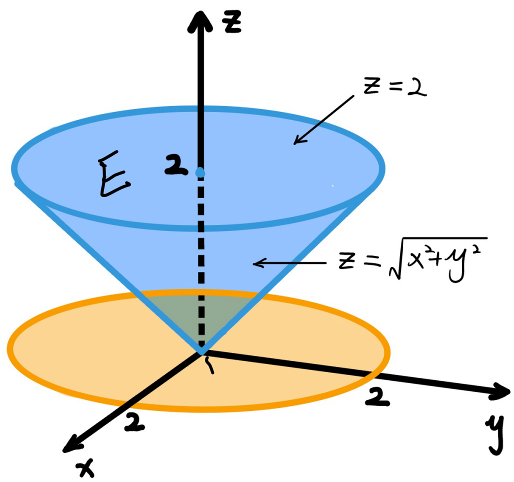
$$\mathrm{Vol}(E) = \iiint_E dV$$
$$= \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$
$$= \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \frac{1}{3} \rho^3 \sin\phi \Big|_{\rho=0}^{\rho=\cos\phi} d\phi \, d\theta$$
$$= \frac{1}{3} \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \cos^3\phi \sin\phi \, d\phi \, d\theta$$
$$= \frac{1}{3} \int_0^{2\pi} \int_0^{\frac{\pi}{4}} -\cos^3\phi \, d\cos\phi \, d\theta$$
$$= \frac{1}{3} \int_0^{2\pi} -\frac{1}{4} \cos^4\phi \Big|_0^{\frac{\pi}{4}} d\theta$$
$$= \frac{1}{3} \left(\int_0^{2\pi} d\theta \right) \left(-\frac{1}{4} \left[\cos^4 \frac{\pi}{4} - \cos^4 0 \right] \right)$$
$$= \frac{2\pi}{3} \left(-\frac{1}{4} \left(\left(\frac{1}{\sqrt{2}} \right)^4 - 1 \right) \right) \quad \left(\tfrac{1}{4}\right)$$
$$= \frac{2\pi}{3} \cdot \frac{3}{16}$$
$$= \frac{\pi}{8}$$

Review on cylindrical coordinates

Example 3. Evaluate the integral by changing to cylindrical coordinates:

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\,dzdydx$$

Below is a figure of the solid region of integration that might help with the calculation.



The region E in terms of rectangle coordinate is

$$\sqrt{x^2+y^2} \le z \le 2$$
$$-\sqrt{4-x^2} \le y \le \sqrt{4-x^2}$$
$$-2 \le x \le 2$$

Note the projection of E onto xy-plane is the disk $x^2+y^2 \le 4$. Thus E is easier to describe in terms of cylindrical coordinate.

$$E: \quad \sqrt{x^2+y^2} = r \le z \le 2$$
$$0 \le \theta \le 2\pi$$
$$0 \le r \le 2$$

Thus $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\,dzdydx$

$$= \iiint_E (x^2+y^2)\,dV$$

$$= \int_0^{2\pi}\int_0^2\int_r^2 r^2\, r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^2\int_r^2 r^3\,dz\,dr$$

$$= 2\pi \cdot \int_0^2 r^3 \cdot z \Big|_{z=r}^{z=2} dr$$

$$= 2\pi \int_0^2 \left(2r^3 - r^4\right) dr$$

$$= 2\pi \left[\frac{1}{2}r^4 - \frac{1}{5}r^5\right]\Big|_0^2$$

$$= 2\pi \left[8 - \frac{32}{5}\right]$$

$$= 2\pi \cdot \frac{8}{5}$$

$$= \frac{16\pi}{5}$$

Change of Variables in Multiple Integrals

Change of Variable in one-dimensional calculus

Recall in calculus, we have the change of the variable formula:

$$\int_a^b f(x)dx = \int_c^d f(g(u))g'(u)du$$

where $x = g(u)$ and $a = g(c), b = g(d)$.

And we can rewrite it as

$$\int_a^b f(x)dx = \int_c^d f(x(u))\frac{dx}{du}du \tag{2}$$

Next, we are going to see the generalization of equation (2) to higher dimensions (these are Equation (7) and (8)).

Change of Variable in double integrals

- A change of variables can also be used in double integrals.
- One example is the conversion to polar coordinates.
- The new variables r and θ are related to the old variables x and y by the equations

  $$x = r\cos\theta \quad y = r\sin\theta$$

  and the change of variables formula can be written as (Lecture 15)

  $$\iint_R f(x,y)dA = \iint_S f(r\cos\theta, r\sin\theta)rdrd\theta,$$

  where S is the region in the $r\theta$-plane that corresponds to the region R in the xy-plane.

- Generally, we consider a change of variables that is given by a vector-valued function T from the uv-plane to the xy-plane:

  $$T(u,v) = (x,y)$$

  where x and y are related to u and v by the equations

  $$x = g(u,v) \quad y = h(u,v) \tag{3}$$

  or, as we sometimes write,

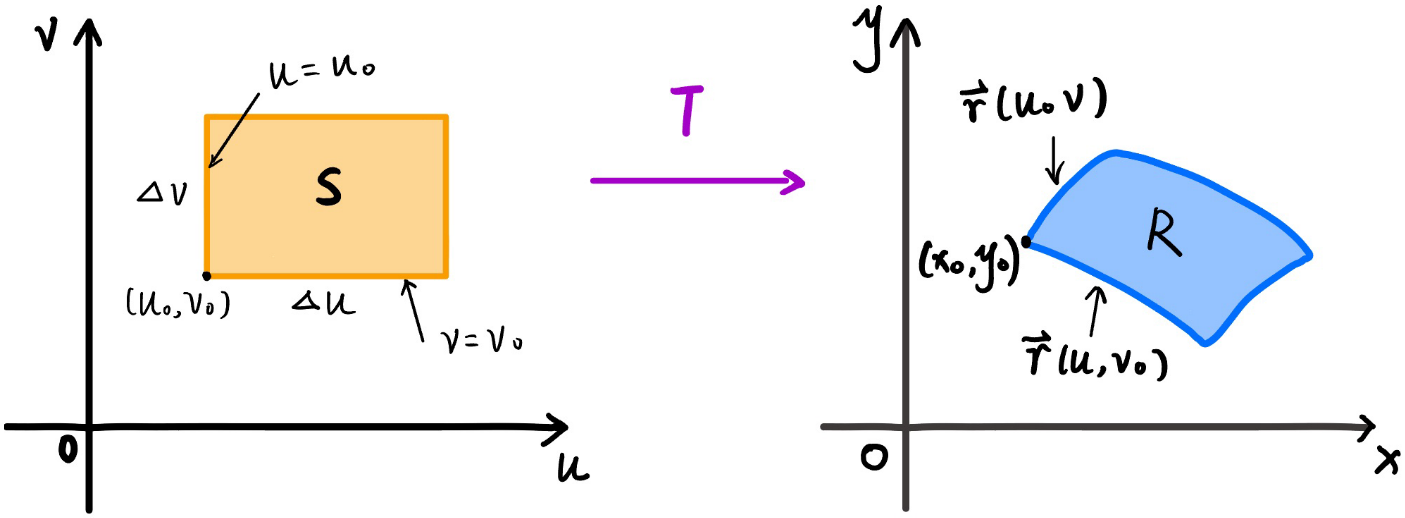
  $$x = x(u,v) \quad y = y(u,v)$$

- We usually assume that T is a C^1 function, which means that g and h have continuous first-order partial derivatives.

- If T is a one-to-one transformation, then it has an inverse transformation T^{-1} from the xy-plane to the uv-plane and it may be possible to solve Equations (3) for u and v in terms of x and y:

$$u = G(x, y) \quad v = H(x, y)$$

- A natural question to ask is **how a change of variables affects a double integral.**
- To answer this question, we look at a small rectangle S in the uv-plane whose lower left corner is the point (u_0, v_0) and whose dimensions are Δu and Δv in the following figure.



- The image of S is a region R in the xy-plane, one of whose boundary points is $(x_0, y_0) = T(u_0, v_0)$.
- The vector

$$\mathbf{r}(u, v) = g(u, v)\mathbf{i} + h(u, v)\mathbf{j}$$

is the position vector of the image of the point (u, v).
- The equation of the lower side of S is $v = v_0$, whose image curve is given by the vector function $\mathbf{r}(u, v_0)$.
- The tangent vector at (x_0, y_0) to this image curve is

$$\mathbf{r}_u = g_u(u_0, v_0)\mathbf{i} + h_u(u_0, v_0)\mathbf{j} = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j}$$

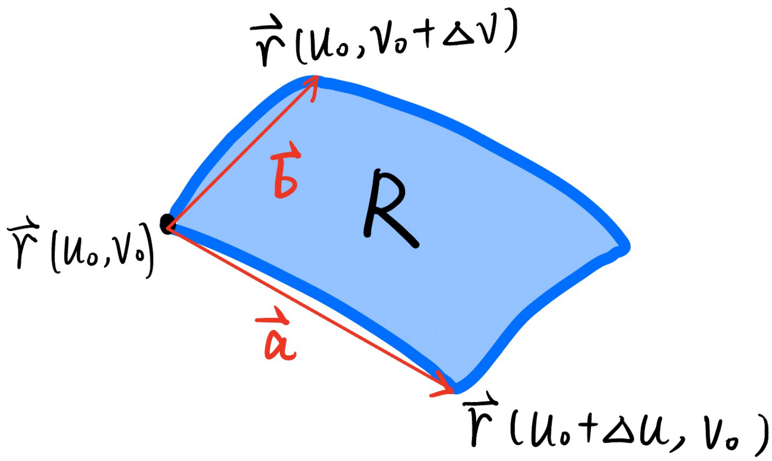
- Similarly, the tangent vector at (x_0, y_0) to the image curve of the left side of S (namely, $u = u_0$) is

$$\mathbf{r}_v = g_v(u_0, v_0)\mathbf{i} + h_v(u_0, v_0)\mathbf{j} = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j}$$

- We can approximate the image region $R = T(S)$ by a parallelogram determined by the secant vectors

$$\mathbf{a} = \mathbf{r}\left(u_0 + \Delta u, v_0\right) - \mathbf{r}\left(u_0, v_0\right) \quad \mathbf{b} = \mathbf{r}\left(u_0, v_0 + \Delta v\right) - \mathbf{r}\left(u_0, v_0\right)$$

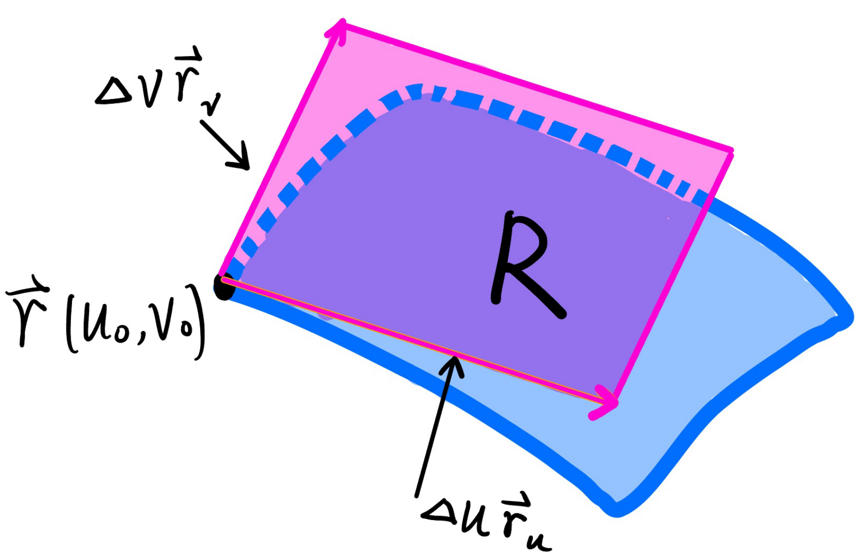
as described in the following figure.



- Note

$$\mathbf{r}_u = \lim_{\Delta u \to 0} \frac{\mathbf{r}\left(u_0 + \Delta u, v_0\right) - \mathbf{r}\left(u_0, v_0\right)}{\Delta u}$$

- Thus
  $\mathbf{r}\left(u_0 + \Delta u, v_0\right) - \mathbf{r}\left(u_0, v_0\right) \approx \Delta u \mathbf{r}_u$
  $\mathbf{r}\left(u_0, v_0 + \Delta v\right) - \mathbf{r}\left(u_0, v_0\right) \approx \Delta v \mathbf{r}_v$
- Therefore, we can approximate R by a parallelogram determined by the vectors $\Delta u \mathbf{r}_u$ and $\Delta v \mathbf{r}_v$ in the following figure.



- Thus we can approximate the area of R by the area of this parallelogram, that is ,

$$\left|\left(\Delta u \mathbf{r}_u\right) \times \left(\Delta v \mathbf{r}_v\right)\right| = \left|\mathbf{r}_u \times \mathbf{r}_v\right| \Delta u \Delta v \tag{4}$$

- Computing the cross product, we get

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \mathbf{k} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \mathbf{k}$$

- Notice the matrix inside of the determinant above is the matrix DT, the derivative of the function of T.
- The above determinant is called the **Jacobian determinant** of T. Note sometimes it is also called the Jacobian in some references.

Definition Jacobian determinant of a Function $T : \mathbb{R}^2 \to \mathbb{R}^2$

The Jacobian determinant of the function T given by $x = g(u, v)$ and $y = h(u, v)$ is

$$\frac{\partial(x,y)}{\partial(u,v)} = \det(DT) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \tag{5}$$

- Then using Equation (4), we have an approximation of the area of R:

$$\Delta A \approx \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \Delta u \Delta v \tag{6}$$

  where the Jacobian determinant is evaluated at (u_0, v_0).

- Next, we consider the image of a small rectangles S_{ij} and call their images in the xy-plane R_{ij} as in the following figure:

- Applying the approximation (6) to each R_{ij}, we approximate the double integral of f over R As:

$$\begin{aligned} \iint_R f(x,y)dA &\approx \sum_{i=1}^{m}\sum_{j=1}^{n} f\left(x_i, y_j\right)\Delta A \\ &\approx \sum_{i=1}^{m}\sum_{j=1}^{n} f\left(g\left(u_i, v_j\right), h\left(u_i, v_j\right)\right) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \Delta u \Delta v \end{aligned}$$

  where the Jacoabian determinant is evaluated at (u_i, v_j).

- Notice that the double sum is a Riemann sum for the integral

$$\iint_S f(g(u,v), h(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dudv$$

The above discussion implies the following theorem:

Theorem 1 Change of Variables in a Double Integral

Let D and D^* be elementary regions in $\mathbb{R}^2$ and let $T : D^* \to D$ be a C^1, one-to-one map such that $T(D^*) = D$. For any integrable function $f : D \to \mathbb{R}$,

$$\iint_D f(x,y)dA = \iint_{D^*} f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dA^*, \tag{7}$$

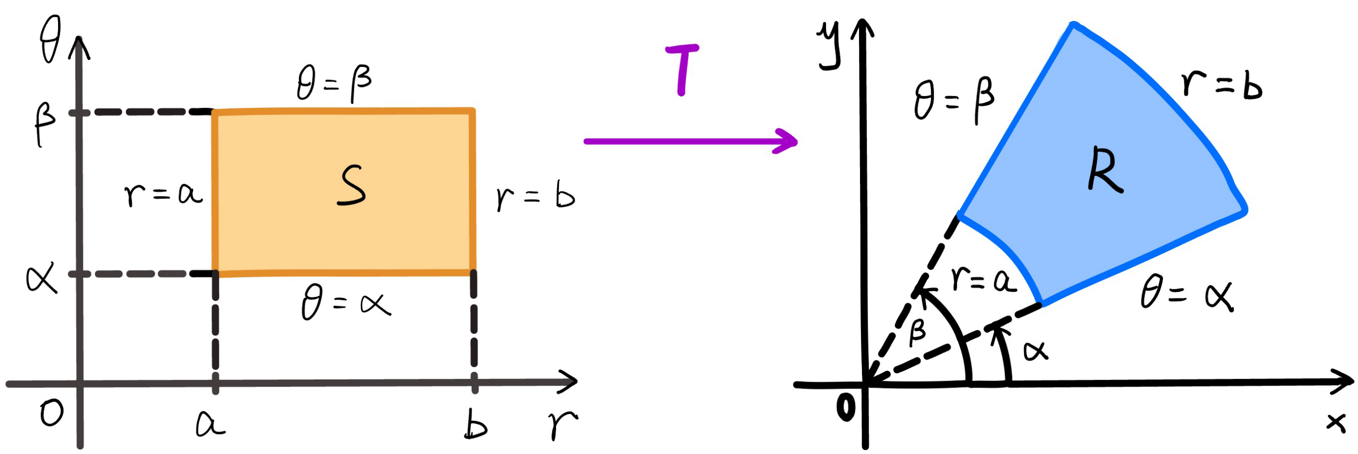
where $\dfrac{\partial(x,y)}{\partial(u,v)}$ is the Jacobian determinant of T.

Application: Revisit the Change to Polar Coordinates in a Double Integral

- We will show that the formula for integration in polar coordinates is a special case of formula (7).
- Here the function T from the $r\theta$-plane to the xy-plane is given by

$$x = g(r,\theta) = r\cos\theta \quad y = h(r,\theta) = r\sin\theta$$

  and the geometry of the transformation is shown below.



- T maps an ordinary rectangle in the $r\theta$-plane to a polar rectangle in the xy-plane.
- The Jacobian determinant of T is

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r > 0$$

- Thus Theorem 1 implies

$$\begin{aligned}\iint_R f(x,y)dxdy &= \iint_S f(r\cos\theta, r\sin\theta)\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right|drd\theta \\ &= \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)rdrd\theta\end{aligned}$$

  which is the same as the formula in Lecture 15.

Change of Variables in Triple Integrals

There is a similar change of variables formula to Theorem 1 for triple integrals.

Let T be a transformation that maps a region S in uvw-space onto a region R in xyz-space by equations

$$x = g(u,v,w) \quad y = h(u,v,w) \quad z = k(u,v,w)$$

The Jacobian determinant of T is the following 3×3 determinant:

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det(DT) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Similar to Theorem 1, we have the following formula for the change of variables in triple integrals:

$$\iiint_R f(x,y,z)dV = \iiint_S f(x(u,v,w), y(u,v,w), z(u,v,w))\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|dudvdw \tag{8}$$

Example 5. Compute the Jacobian for the change of variables into spherical coordinates:

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi$$

Then use Formula (8) to derive the formula for triple integration in spherical coordinates.

Solution.

We compute the Jacobian determinate as follows:

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix}$$

Expanding the determinate in terms of the 3rd row, we get

$$\begin{aligned} &\cos\phi \begin{vmatrix} -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \end{vmatrix} - \rho\sin\phi \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix} \\ &= \cos\phi\left(-\rho^2\sin\phi\cos\phi\sin^2\theta - \rho^2\sin\phi\cos\phi\cos^2\theta\right) \\ &\quad -\rho\sin\phi\left(\rho\sin^2\phi\cos^2\theta + \rho\sin^2\phi\sin^2\theta\right) \end{aligned}$$

Since $0 \leqslant \phi \leqslant \pi$, we have $\sin\phi \geqslant 0$. Therefore

$$\left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}\right| = \left|-\rho^2\sin\phi\right| = \rho^2\sin\phi$$

and Formula (8) implies

$$\iiint_R f(x,y,z)dV = \iiint_S f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi d\rho d\theta d\phi$$

which is equivalent to Formula (1).